MMAT5510 Foundation of Advanced Mathematics
Assignment 1
Suggested solution

1.

$$\begin{aligned} & x \in A \wedge \big(\neg(x \in B \cup C)\big) \\ \equiv\ & x \in A \wedge \big(\neg(x \in B) \wedge \neg(x \in C)\big) \\ \equiv\ & \big(x \in A \wedge \neg(x \in B)\big) \wedge \neg(x \in C) \\ \equiv\ & x \in A\backslash B \ \wedge\ \neg(x \in C) \end{aligned}$$

Let $A = \{1, 2, 3, 4\}$, $B = \{1, 3\}$, $C = \{1\}$
Then

$$A\backslash(B\backslash C) = \{1, 2, 4\}, \qquad (A\backslash B)\backslash C = \{2, 4\}.$$

The set $A\backslash(B\backslash C)$ contains the element 1 but the set $(A\backslash B)\backslash C$ does not.
Therefore $A\backslash(B\backslash C)$ and $(A\backslash B)\backslash C$ are not the same.

2. Let l be the least common multiple (l.c.m.) of m and n.
Let C be a set of integers which are divisible by l, i.e.

$$C = \{x \in \mathbb{Z} \ :\ x \text{ is divisible by } l\}.$$

We claim that $A \cap B = C$.

Let $x \in A \cap B$, then x is divisible by both m and n. By division algorithm, we have $x = yl + z$ for some integer y and z with $0 \le z < l$. Since x, yl are divisible by m, z is also divisible by m. Similarly, z is also divisible by n. We must have $z = 0$ otherwise we get a contradiction with the definition of l.c.m. of m and n. Hence $x \in C$.
$\therefore A \cap B \subseteq C$.

Let $x \in C$. Since x is divisible by l and l is divisible by both m and n, x is divisible by both m and n. So $x \in A \cap B$.
$\therefore C \subseteq A \cap B$.

3. (Prove by contradiction)
Let x be a rational number and y be an irrational number. Let $z = x + y$.

Suppose z is a rational number. Then $z = \frac{m}{n}$ for some integers m and n with $n \neq 0$ and $x = \frac{r}{s}$ for some integers r and s with $s \neq 0$ as x is rational number. We then have

$$y = z - x = \frac{m}{n} - \frac{r}{s} = \frac{ms - rn}{ns}.$$

Both $ms - rn$ and ns are integers with $ns \neq 0$, so y is rational number.
Contradiction arises.

4. (Prove by contrapositive)

Suppose that n is not odd,
then n is even and $n = 2m$ for some integer m.

Then $n^2 = 4m^2 = 2(2m^2)$ where $2m^2$ is an integer.
Then n^2 is even.
Therefore, if n^2 is odd, then n is odd.

5a. i (Reflexive) If $(x, y) \in \mathbb{R}^2$, then $(x, y) \sim (x, y)$ since $y - y = x - x = 0$.

ii (Symmetric) If (x, y), $(m, n) \in \mathbb{R}^2$ and $(x, y) \sim (m, n)$, then $n - y = m - x$. This implies $y - n = x - m$.
$\therefore (m, n) \sim (x, y)$.

iii (Transitive) If (x, y), (m, n), $(r, s) \in \mathbb{R}^2$, $(x, y) \sim (m, n)$ and $(m, n) \sim (r, s)$, then $n - y = m - x$ and $s - n = r - m$.

$$\begin{aligned} s - y &= (s - n) + (n - y) \\ &= (r - m) + (m - x) \\ &= r - x \end{aligned}$$

$\therefore (x, y) \sim (r, s)$.

Therefore, $\sim$ is an equivalence relation on $\mathbb{R}^2$.

5b. $(0, 0) \sim (x, y)$ if and only if $y - 0 = x - 0$, i.e $x = y$.
Therefore,

$$[(0, 0)] = \{(x, y) \in \mathbb{R}^2 \ : \ x = y\}.$$

6a. i (Reflexive) If $P(x) \in \mathbb{R}[x]$, then $P(x) \sim P(x)$ since $P(x) - P(x) = 0$ which is divisible by $x - 1$.

ii (Symmetric) If $P(x)$, $Q(x) \in \mathbb{R}[x]$ and $P(x) \sim Q(x)$, then $P(x) - Q(x)$ is divisible by $x - 1$, i.e. $P(x) - Q(x) = (x - 1)f(x)$ for some $f(x)$ in $\mathbb{R}[x]$.
Then $Q(x) - P(x) = (x - 1)(-f(x))$ which is divisible by $x - 1$.
$\therefore Q(x) \sim P(x)$.

iii (Transitive) If $P(x)$, $Q(x)$, $R(x) \in \mathbb{R}[x]$, $P(x) \sim Q(x)$ and $Q(x) \sim R(x)$, then $P(x) - Q(x)$ is divisible by $x - 1$ and $P(x) - Q(x) = (x - 1)f(x)$ for some $f(x)$ in $\mathbb{R}[x]$.
$Q(x) - R(x)$ is divisible by $x - 1$ and $Q(x) - R(x) = (x - 1)g(x)$ for some $g(x)$ in $\mathbb{R}[x]$.

$$\begin{aligned} \text{Hence } \ P(x) - R(x) &= (P(x) - Q(x)) + (Q(x) - R(x)) \\ &= (x - 1)f(x) + (x - 1)g(x) \\ &= (x - 1)(f(x) + g(x)) \text{ which is divisible by } x - 1. \end{aligned}$$

$\therefore P(x) \sim R(x)$.

Therefore, $\sim$ is an equivalence relation on $\mathbb{R}[x]$.

6b. $P(x) \sim 2$ if and only if $P(x) - 2$ is divisible by $x - 1$, i.e. $P(x) - 2 = (x - 1)Q(x)$ for some $Q(x)$ in $\mathbb{R}[x]$.
Therefore,

$$[2] = \{P(x) \in \mathbb{R}[x] \ : \ P(x) = (x - 1)Q(x) + 2 \text{ for some } Q(x) \text{ in } \mathbb{R}[x]\}.$$

7a. Let (m,n), (m',n'), (p,q), $(p',q') \in \mathbb{N}^2$ such that $(m,n) \sim (m',n')$ and $(p,q) \sim (p',q')$.
Then $m+n' = m'+n$ and $p+q' = p'+q$.

$$\begin{aligned}(m,n)*(p,q) &= (m\cdot p + n\cdot q, n\cdot p + m\cdot q)\\ (m',n')*(p',q') &= (m'\cdot p' + n'\cdot q', n'\cdot p' + m'\cdot q')\end{aligned}$$

$$\begin{aligned}m\cdot p + n\cdot q + m'\cdot q' + n'\cdot p' + m\cdot q' &= m\cdot p' + n\cdot q + m'\cdot q' + n'\cdot p' + m\cdot q\\ &= m'\cdot p' + n\cdot q + m'\cdot q' + n\cdot p' + m\cdot q\\ &= m'\cdot p' + n\cdot q' + m'\cdot q' + n\cdot p + m\cdot q\\ &= m'\cdot p' + n'\cdot q' + m\cdot q' + n\cdot p + m\cdot q\end{aligned}$$

Since $m\cdot p + n\cdot q + m'\cdot q' + n'\cdot p' + m\cdot q' = m\cdot q + n\cdot p + m'\cdot p' + n'\cdot q' + m\cdot q'$, we have $m\cdot p + n\cdot q + m'\cdot q' + n'\cdot p' = m\cdot q + n\cdot p + m'\cdot p' + n'\cdot q'$.
$\therefore (m,n)*(p,q) \sim (m',n')*(p',q')$.
Therefore, the multiplication $*$ on $\mathbb{N}^2$ induces a multiplication on $\mathbb{Z}$.

7b Let $[(m,n)]$, $[(p,q)]$, $[(r,s)] \in \mathbb{Z}$ where (m,n), (p,q), $(r,s) \in \mathbb{N}^2$.
(Commutative)

$$\begin{aligned}[(m,n)]*[(p,q)] &= [(m,n)*(p,q)]\\ &= [(m\cdot p + n\cdot q, n\cdot p + m\cdot q)]\\ &= [(p\cdot m + q\cdot n, q\cdot m + p\cdot n)]\\ &= [(p,q)*(m,n)]\\ &= [(p,q)]*[(m,n)]\end{aligned}$$

(Associative)

$$\begin{aligned}&\big([(m,n)]*[(p,q)]\big)*[(r,s)]\\ =&[(m\cdot p + n\cdot q, n\cdot p + m\cdot q)]*[(r,s)]\\ =&[\big((m\cdot p + n\cdot q)\cdot r + (n\cdot p + m\cdot q)\cdot s, (n\cdot p + m\cdot q)\cdot r + (m\cdot p + n\cdot q)\cdot s\big)]\\ =&[\big(m\cdot(p\cdot r + q\cdot s) + n\cdot(q\cdot r + p\cdot s), n\cdot(p\cdot r + q\cdot s) + m\cdot(q\cdot r + p\cdot s)\big)]\\ =&[(m,n)]*[(p\cdot r + q\cdot s, q\cdot r + p\cdot s)]\\ =&[(m,n)]*\big([(p,q)]*[(r,s)]\big)\end{aligned}$$

8. i (Reflexive) If $f \in C^\infty$, then $f \sim f$ since $f(a) = f(a)$ and $f'(a) = f'(a)$.

ii (Symmetric) If $f, g \in C^\infty$ and $f \sim g$, then $f(a) = g(a)$ and $f'(a) = g'(a)$ which means $g(a) = f(a)$ and $g'(a) = f'(a)$.
$\therefore g \sim f$.

iii (Transitive) If $f, g, h \in C^\infty$, $f \sim g$ and $g \sim h$, then $f(a) = g(a)$, $f'(a) = g'(a)$, $g(a) = h(a)$ and $g'(a) = h'(a)$.
Then $f(a) = g(a) = h(a)$ and $f'(a) = g'(a) = h'(a)$.
$\therefore f \sim h$.

Therefore, $\sim$ is an equivalence relation on C^∞.

9. 'For all x, Not $P(x)$ or (There exists y such that $P(y)$ but $y \neq x$).'
$\forall\ x, \neg P(x) \vee \big(\exists y, P(y) \wedge (y \neq x)\big)$.